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Hadamard's Theorem for Locally Lipschitzian Maps

B. H. POURCIAU

Mathematics Department
Lawrence University, Appleton, Wisconsin 54912

Submitted by Jane Cronin

A result of Hadamard lists conditions forcing a continuously differentiable map $f: R^n \to R^n$ to be a homeomorphism onto R^n:

HADAMARD'S THEOREM. *Let $M > 0$. If the derivative $f'(p)$ is invertible and $\|f'(p)^{-1}\| \leqslant M$ for each p in R^n, then f is a homeomorphism onto R^n.*

In this paper we suppose f is only locally Lipschitzian and search for analogous conditions. Replacing the derivative $f'(p)$ (which now may not exist on a set of measure zero) with a set-valued derivative $\partial f(p)$, we prove this result:

THEOREM. *Let $M > 0$. If A is invertible and $\|A^{-1}\| \leqslant M$ for each linear map A in $\partial f(p)$ and each p in R^n, then f is a homeomorphism onto R^n.*

1. INTRODUCTION

If the derivative $f'(p)$ of a continuously differentiable map f from R^n into itself is an invertible linear transformation for each p, then by the classical inverse function theorem f is a local homeomorphism: every p has a neighborhood U mapped homeomorphically onto its image $f(U)$. A natural and intriguing question—under what further conditions will f be a homeomorphism of R^n onto itself—was answered in one way by Hadamard [8] in 1906, using a condition on the norms $\|B\| = \max\{|Bh|: |h| = 1\}$ of the inverses $B = f'(p)^{-1}$:

HADAMARD'S THEOREM. *Suppose f is a continuously differentiable map from R^n into R^n and let $M > 0$. If $f'(p)$ is invertible and $\|f'(p)^{-1}\| \leqslant M$ for each p in R^n, then f is a homeomorphism from R^n onto R^n.*

What happens to this state of affairs if we weaken the smoothness hypothesis on f? Can we still pose a similar question, and if so, is there a Hadamard-like answer? We shall, in fact, face this problem when f is locally

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Lipschitzian, that is, when every p has a neighborhood where f is Lipschitz continuous. (All continuously differentiable maps are locally Lipschitzian.) In this case, there may be points p where the derivative $f'(p)$ does not exist, a malady which motivates our use of a set-valued derivative. To each p we shall assign a certain collection $\partial f(p)$ of linear transformations from R^n into R^n called the generalized derivative at p. Now this notion of differentiability has many nice properties, as we shall see, but only two need to be singled out in this introduction. For one thing, the collection $\partial f(p)$ reduces to the singleton $\{f'(p)\}$ whenever f is continuously differentiable on a neighborhood of p. Theorems involving the generalized derivative therefore extend their continuously differentiable counterparts. The other property we wish to mention, an extension of the classical inverse function theorem, provides an example: if the generalized derivative $\partial f(p)$ of a locally Lipschitzian map f is invertible (that is, if each linear transformation A in $\partial f(p)$ is invertible) at every p, then f is a local homeomorphism.

Naturally this prompts us to ask the question we asked before—under what further conditions must f be a homeomorphism of R^n onto itself—and our answer is just what one might hope for:

LIPSCHITZIAN HADAMARD THEOREM. *Suppose f is a locally Lipschitzian map from R^n into R^n and let $M > 0$. If $\partial f(p)$ is invertible and $\|A^{-1}\| \leqslant M$ for each p in R^n and each A in $\partial f(p)$, then f is a homeomorphism from R^n onto R^n.*

After discussing the generalized derivative in Section 2, we shall prove this theorem in Section 3.

2. THE GENERALIZED DERIVATIVE

Let R^n be real euclidean n-space and let $L(R^n, R^k)$ be the space of linear transformations from R^n into R^k. Whenever $x \in R^n$, $A \in L(R^n, R^k)$, $S \subset R^n$, and $\delta > 0$, then $|x|$ denotes the euclidean norm of x, $\|A\|$ the familiar norm $\max\{|Ax| : |x| = 1\}$ of A, $B_\delta(x)$ the δ-ball $\{z \in R^n : |z - x| < \delta\}$ about x, and $B_\delta(S)$ the δ-neighborhood $\bigcup \{B_\delta(z) : z \in S\}$ of S. The convex hull, closure of the convex hull, and (when S is measurable) the n-dimensional Lebesgue measure of S will be denoted by $\operatorname{co} S$, $\overline{\operatorname{co}}\, S$, and $\mu(s)$, respectively.

With the notation set, we can move to the definition of the generalized derivative. Suppose f maps R^n into R^k. We say f is locally Lipschitzian provided each point x has a neighborhood U where some number M satisfies $|f(z_1) - f(z_2)| \leqslant M |z_1 - z_2|$ for all z_1 and z_2 in U. In this case, by a deep theorem of Rademacher, the (Frèchet) derivative $f'(x)$ exists at μ-almost every x. (See Frederer [6].) Moreover, μ-almost every x is a Lebesgue point of the derived mapping f'. By definition such x satisfy

$$\lim_{\epsilon \to 0} \frac{1}{\mu B_\epsilon(x)} \int_{B_\epsilon(x)} \|f'(z) - f'(x)\| \, d\mu(z) = 0.$$

Let $L(f')$ stand for the set of all these Lebesgue points and let p belong to R^n. Then the generalized derivative $\partial f(p)$ of f at p is the nonempty, compact, convex subset

$$\bigcap_{\delta > 0} \Delta_\delta(f, p)$$

of $L(R^n, R^k)$, where $\Delta_\delta(f, p)$ denotes the collection

$$\overline{\text{co}}\{f'(x) : x \in B_\delta(p) \cap L(f')\}.$$

Clarke, in [2], introduced the generalized derivative (without the Lebesgue point restriction) to study nonlinear optimization problems with nonsmooth data. Since then, Clarke and others have continued to work with various generalized derivatives to extend results in nonlinear programming, the calculus of variations, optimal control theory, and differential equations. See, for instance, Auslander [1], Clarke [3, 4, 5], Goldstine [7], Halkin [9–11], Hiriart–Urruty [12–14], Penot [16], Pourciau [18–20], Sweetzer [21], and Warga [22–24]. In the present work we use Clarke's generalized derivative with a Lebesgue point condition. This added condition allows us to ignore null sets in forming the generalized derivative. For more on this, see Pourciau [19].

The generalized derivative defined above enjoys many nice properties; many are proved in Pourciau [19]. For the proof of our Hadamard theorem, two basic results are especially important. In the calculus of differentiable functions the Mean Value Theorem is fundamental; in the calculus of locally Lipschitzian maps we have a parallel, due to Lebourg [15] and Pourciau [18]. To write down this result, we need to define the generalized derivative of a segment. Given any points p and q in R^n, let $[p, q]$ stand for the line segment $\{(1-t)p + tq : 0 \leqslant t \leqslant 1\}$. Whenever f is locally Lipschitzian on a neighborhood of $[p, q]$, we abbreviate the collection

$$\text{co} \bigcup \{\partial f(x) : x \in [p, q]\}$$

by $\partial f([p, q])$ and call it the generalized derivative of the segment $[p, q]$. The collection $\partial f([p, q])$ is nonempty, convex, and compact. Now we can record the

MEAN VALUE THEOREM. *Suppose U is an open subset of R^n, f is a locally Lipschitzian map from U into R^k, and $[p, q] \subset U$. Then $f(q) - f(p) = A(q - p)$ for some A in $\partial f([p, q])$.*

The second basic result we need to record here is a Lipschitzian version of the classical inverse function theorem. Consult Clarke [4] or Pourciau [19] for the proof.

INVERSE MAP THEOREM. *Let $U \subset R^n$ be a neighborhood of p, and suppose $f: U \to R^n$ is Lipschitzian, with an invertible generalized derivative $\partial f(p)$. Then p has a neighborhood $V \subset U$ that f maps homeomorphically onto its image $f(V)$. The local inverse g is Lipschitzian on this image, and*

$$g'[f(x)] = f'(x)^{-1}$$

for μ-almost every x in V.

3. THE LIPSCHITZIAN HADAMARD THEOREM

Let us begin the proof of the Lipschitzian Hadamard Theorem with some strategy. Suppose f maps R^n into itself. We say f lifts line segments if for any c in the image $f(R^n)$, any d in R^n, and any a in $f^{-1}(c)$ there is a continuous $\alpha: [0, 1] \to R^n$ satisfying $\alpha(0) = a$ and $f \circ \alpha = \beta$ on $[0, 1]$, where $\beta(t) = (1 - t)c + td$. Plastock [17] has shown the pair (R^n, f) is a covering space for R^n when and only when f is a local homeomorphism and lifts line segments. As a corollary, a local homeomorphism is a homeomorphism onto R^n provided it lifts line segments. But a map f enjoying the hypotheses of the Lipschitzian Hadamard Theorem must be a local homeomorphism, by the Inverse Map Theorem above, so to demonstrate that f carries R^n homeomorphically onto itself, we need only prove that f lifts line segments.

Choose any c in $f(R^n)$, any d in R^n, and suppose $\beta: [0, 1] \to R^n$ is the segment $\beta(t) = (1 - t)c + td$. As a local homeomorphism, f must lift at least an initial piece of β: there must be a number $\bar{s}$ in $(0, 1)$ and a continuous map $\alpha: [0, \bar{s}) \to R^n$ satisfying $f \circ \alpha = \beta$ on $[0, \bar{s})$. In fact, setting $\alpha = g \circ \beta$, where g is the local inverse of f about $c = \beta(0)$, provides a suitable α. Let S stand for the least upper bound of those numbers s in $[0, 1]$ such that α extends to a continuous map on $[0, s)$ with $f \circ \alpha = \beta$.

For any t in $[0, S)$, if g is the local Lipschitzian inverse of f defined on a neighborhood of $\beta(t)$, then $\alpha = g \circ \beta$ on a neighborhood of t, so α is locally Lipschitzian on $[0, S)$. If α is actually Lipschitzian on $[0, S)$, that is, if α has a uniform Lipschitz constant on $[0, S)$, we can show f lifts line segments. For in this case, the limit $\bar{a} = \lim_{t \to S^-} \alpha(t)$ exists (R^n is complete), and putting $\alpha(S) = \bar{a}$ makes α continuous and satisfy $f \circ \alpha = \beta$ on $[0, S]$. But since f is a local homeomorphism, α would then extend to some $t > S$, contradicting the definition of S, unless $S = 1$.

So, to finish the proof, we must show α is Lipschitzian on $[0, S)$. Assume, for the moment, there is a constant K such that for every t in $[0, S)$ and every A in $\partial\alpha(t)$, we have $\|A\| \leqslant K$. Then surely for every t' and t'' in $[0, S)$ and every A in

$$\partial\alpha([t', t'']) = \operatorname{co} \bigcup \{\partial\alpha(t) \colon t \in [t', t'']\}$$

we have $\|A\| \leqslant K$. But for each such pair t', t'' the Mean Value Theorem implies $\alpha(t'') - \alpha(t') = A(t'' - t')$ for some A in $\partial\alpha([t', t''])$, and therefore K is a uniform Lipschitz constant for α on $[0, S)$.

We are thus reduced to finding a constant K such that for all t in $[0, S)$ and all A in $\partial\alpha(t)$, we have $\|A\| \leqslant K$. Choose any t in $[0, S)$. Let V be the neighborhood of $\alpha(t)$ and g the local Lipschitzian inverse advertised in the Inverse Map Theorem. Then $\alpha = g \circ \beta$ on some neighborhood of t. We have a chain rule for generalized derivatives (consult Clarke [3] or Pourciau [19] for the proof), and it implies

$$\partial\alpha(t)h \subset \partial g[\beta(t)] \circ \beta'(t)h \tag{*}$$

for every real number h. (The right side of the inclusion is the set of all compositions.) Let us then study the collection $\partial g[\beta(t)]$. By definition,

$$\partial g[\beta(t)] = \bigcap_{\delta > 0} \overline{\operatorname{co}}\{g'(y) \colon y \in B_\delta[\beta(t)] \cap L(g')\}.$$

Whenever δ is sufficiently small, say $0 < \delta \leqslant \bar{\delta}$, the image of V under f contains $B_\delta[\beta(t)]$. Now by the Inverse Map Theorem, the set

$$N = \{x \in V \colon \text{either } f'(x) \text{ or } g'[f(x)] \text{ does not exist}\}$$

has μ-measure 0. Since locally Lipschitzian maps carry null sets onto null sets, $f(N)$ also has μ-measure 0, and this means μ-almost every y in $B_{\bar{\delta}}[\beta(t)]$ satisfies $y = f(x)$ for some x in V and $g'[f(x)] = f'(x)^{-1}$. Let Y stand for the set of all such y. Recall the Lebesgue point restriction in our definition of generalized derivative permits us to ignore sets of μ-measure 0. (See Pourciau [19], Proposition 4.2.) This means

$$\partial g[\beta(t)] = \bigcap_{0 < \delta < \bar{\delta}} \overline{\operatorname{co}}\{g'(y) \colon y \in B_\delta[\beta(t)] \cap L(g') \cap Y\}.$$

Yet by hypothesis there is a number M such that $\|f'(x)^{-1}\| \leqslant M$ for every x in R^n where $f'(x)$ exists, and it follows that $\|B\| \leqslant M$ for every B in $\partial g[\beta(t)]$.

Noting that M is independent of t and recalling the inclusion (*), we infer $\|A\| \leqslant M|d-c|$ whenever $A \in \partial\alpha(t)$ and $t \in [0, S)$. Thus $M|d-c|$ is the constant K we needed, and the proof is complete.

REFERENCES

1. A. AUSLANDER, Minimisation sans contraintes de fonctions localement Lipschitziennes: Application à la programmation mi-convexe, midifférentiable, *C. R. Acad. Sci. Paris, Sér. A* **284** (1977), 959–961, and "Proceedings of the 3rd Symposium on Nonlinear Programming," University of Wisconsin, Madison.
2. F. H. CLARKE, "Necessary Conditions for Nonsmooth Problems in Optimal Control and Calculus of Variations," Ph.D. dissertation, University of Washington, 1973.
3. F. H. CLARKE, Generalized gradients and applications, *Trans. Amer. Math. Soc.* **205** (1975), 247–262.
4. F. H. CLARKE, On the inverse function theorem, *Pacific J. Math.* **64**, No. 1 (1976), 97–102.
5. F. H. CLARKE, A new approach to Lagrange multipliers, *Math. Oper. Res.* **2** (1976), 165–174.
6. H. FEDERER, "Geometric Measure Theory," Springer-Verlag, Berlin/New York, 1969.
7. A. GOLDSTINE, Optimization of Lipschitz continuous functions, *Math. Program.* **13** (1977), 14–22.
8. J. HADAMARD, Sur les transformations ponctuelles, *Bull. Soc. Math. Fr.* **34**, 71–84.
9. H. HALKIN, Interior mapping theorem with set-valued derivatives, *J. Anal. Math.* **30** (1976), 200–207.
10. H. HALKIN, Mathematical programming without differentiability, *in* "Calculus of Variations and Control Theory" (D. Russell, Ed.), Academic Press, New York, 1976.
11. H. HALKIN, Optimization without differentiability, to appear.
12. J-B HIRIART-URRUTY, On necessary optimality conditions in nondifferentiable programming, *Math. Program.* **14**, No. 1 (1978), 73–86.
13. J-B HIRIART-URRUTY, Tangent cones, generalized gradients and mathematical programming in Banach spaces, *Math. Oper. Res.* (1978).
14. J-B HIRIART-URRUTY, Refinements of necessary optimality conditions in nondifferentiable programming, I, to appear.
15. F. LEBOURG, Valeur moyenne pour gradient généralisé, *C. R. Acad. Sci. Paris, Sér. A* **10** (1975).
16. J-P PENOT, Calcul sons-differentiel et optimisation, *J. Funct. Anal.* **27** (1978), 248–276.
17. R. PLASTOCK, Homeomorphisms between Banach spaces, *Trans. Amer. Math. Soc.* **200**, 169–183.
18. B. H. POURCIAU, "A Generalized Derivative and Its Applications to Optimization Theory, Classical Analysis, and Global Univalence," Ph.D. dissertation, University of California, 1976.
19. B. H. POURCIAU, Analysis and optimization of Lipschitz continuous mappings, *J. Optim. Theory Appl.* **22**, No. 3 (1977), 311–351.
20. B. H. POURCIAU, Univalence and degree for Lipschitz continuous maps, *Arch. Rat. Mech. Anal.*, in press.
21. T. H. SWEETSER, A minimal set-valued strong derivative for vector valued Lipschitz functions, *J. Optim. Theory Appl.* **23**, No. 4 (1977), 549–562.
22. J. WARGA, Necessary conditions without differentiability in optimal control, *J. Differential Equations*, in press.

23. J. WARGA, Derivate containers, inverse functions and controllability, *in* "Proceedings of MRC Symposium on the Calculus of Variations and Optimal Control, Sept. 1975" (D. L. Russel, Ed.), Academic Press, New York, 1976.
24. J. WARGA, "Optimal Control of Differential and Functional Equations," Academic Press, New York, 1972.

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